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BOOK OF ABSTRACTS

Miura maps for Stäckel systems

Maciej Błaszak

Adam Mickiewicz University Poznań, Poland

In this talk first I define a Miura map for finite dimensional Hamiltonian systems. Then, I construct a family of such maps for particular class of Stäckel systems. Finally, I show that these maps lead to multi-Hamiltonian representation of considered systems.

Discrete gradient numerical methods for Hamiltonian and dissipative dynamical systems

Jan L. Cieśliński and Artur Kobus

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Discrete gradient numerial methods were developed to solve numerically problems in molecular dynamics or n-body problems [1]. The main idea is surprisingly simple and can be easily illustrated in the case of the one-dimensional Newton equation ($\dot{p} = -\partial V/\partial x$, $p = \dot{x}$, where the potential $V = V(x)$ is given) which can be discretized as follows:

$$\frac{p_{n+1} - p_n}{h} = -\frac{V(x_{n+1}) - V(x_n)}{x_{n+1} - x_n} \, , \qquad \frac{p_{n+1} + p_n}{2} = \frac{x_{n+1} - x_n}{h} \, .$$

Multiplying both equations side by side we got the exact conservation of the energy integral. This approach was generalized on cases with any number of first integrals and Lyapunov functions (i.e., functions decreasing with time) in the seminal paper [2].

We show that the discrete gradient methods can be improved in two different ways without losing the energy-preserving property. First, we can construct modifications of arbitrary high order [3, 4]. Second, we introduce so called locally exact discretizations which are extremely accurate in the neighbourhood (not so small, in fact) of stable equilibria [5, 6].

Our recent results concern dissipative systems [7], including those with non-linear damping (like van der Pol oscilaltor or the inflaton model in cosmology). Surprisingly enough, it seems that the discrete gradient approach is applicable to a very large class of "para-hamiltonian" dynamical systems [8]. Integrals of motions are helpful but not necessary.

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How to integrate an integrable 2 degrees of freedom Hamiltonian?

Thierry Combot

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Consider a rational Hamiltonian H admitting an independant rational first integral I. The phase space is foliated by algebraic Liouville tori of dimension 2. If H is not superintegrable, are these tori exactly common levels of H, I for a suitable choice of I? Once a foliation of irreducible invariant surfaces is obtained, we will distinguish the problem of global integration and local integration, depending on the Galois group of the Picard Fuchs connection of the foliation. Then the integration problem can be expressed in terms of the Albanese morphism of a Liouville torus. The notion of "Algebraic complete integrability" uses morphisms to Jacobian varieties, and we will generalize it using also ruled surfaces. We will see that some new and old systems can be integrated this way.

Improved procedure of semi-classical quantisation of 3-particles Toda lattice

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Usual approaches to quantisation of a 3-particle Toda system lead to sophisticated numerical calculations. Such calculations require many time consuming steps are usually preformed on supercomputers/clusters. We propose a reformulation of the mathematical framework of the EKB quantisation procedure with more adapted variables that reduces extremely the numerical part and eases subsequent numerical calculations. The resulting equations and procedure might be easily implemented in a short Mathematica code.

A time-dependent energy-momentum method

Javier de Lucas Araujo

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First, I will briefly describe the Marsden-Weinstein reduction of time-dependent Hamiltonian systems on symplectic manifolds. In short, this reduction allows one to study a time-dependent Hamiltonian system admitting a Lie group of Hamiltonian symmetries via a time-dependent Hamiltonian system on a symplectic manifold of smaller dimension, the so-called reduced time-dependent Hamiltonian system, which arises by "skipping certain variables" of the initial one.

A reduced time-dependent Hamiltonian system may have equilibrium points that do not necessarily come from equilibrium points of its initial time-dependent Hamiltonian system. We will call such points relative equilibrium points. They are relevant, for instance, in the study of rigid bodies. More particularly, it is interesting to study the stability around equilibrium points of reduced time-dependent Hamiltonian systems, i.e. whether solutions close to an equilibrium point tend to or move away from it.

The energy-momentum method studies the stability close to equilibrium points of a reduced Hamiltonian system through its original Hamiltonian system, but it only applies to time-independent Hamiltonian systems. To solve this drawback, I will introduce a time-dependent energy-momentum method, which also entails the geometric generalisation to a time-dependent setting of stability techniques. Our results allow for studying the motion of spinning ballet dancers, the analysis of the formation and motion of planets, the motion of tennis rackets, and other relevant and/or interesting classical mechanical systems, like the so-called 'almost'-rigid bodies, whose shape may vary on time in a manner so that they can be described by means of a time-dependent inertia tensor.

Tangent lifts of bi-Hamiltonian structures

Alina Dobrogowska

University of Białystok, Poland

We construct several Poisson structures on the tangent bundle TM to a Poisson manifold M using the Lie algebroid structure on the cotangent bundle T^*M. We also show that bi-Hamiltonian structure from M can be transferred to its tangent bundle TM. Moreover, we present how to find Casimir functions for those Poisson structures and we discuss some particular examples.

Non-commutative Hermite-Padé approximation and integrability

Adam Doliwa

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Approximation by rational functions (the Pade approximation) proves to be very useful in numerical calculations and works especially well for functions with singularities. It turns out that the nominators an the denominators of the approximants satisfy integrable equations of the Toda lattice family, with discrete variables being the degrees of the polynomials. Hermite used multiseries generalization of the theory to prove transcendency of the Euler constant. I will present a fully non-commutative version of the Hermite-Pade approximants. The approximants satisfy an integrable multifield generalization of the non-commutative discrete-time Toda system.

Deformation quantization on the cotangent bundle of a Lie group

Ziemowit Domański

Poznań University of Technology, Poland

A complete theory of non-formal deformation quantization on the cotangent bundle of a Lie group is presented. The starting point of the construction is a pre-C^*-algebra of observables being a deformation of the classical pre-C^*-algebra of observables. The deformed product from this algebra (the star-product) is introduced by means of an appropriate integral formula. Basic properties of the star-product are proved and the pre-C^*-algebra is extended to a Hilbert algebra and an algebra of distributions. The algebra of distributions contains functions representing all observables interesting from a physical point of view. An operator representation in position space is constructed. The theory is illustrated on the example of the quantization of a rigid body.

Conformal geodesics can not spiral

Maciej Dunajski

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A curve can be said to spiral at a point p if it enters and remains in every neighbourhood of p but does not pass through p itself. It is a classical result, based on the existence of geodesically-convex neighbourhoods that metric geodesics cannot spiral. It has, until now, been an open problem whether the spiraling can occur for conformal geodesics (a preferred set of curves on a manifold with a conformal structure). I will state a recent result which settles this question.

Detailed description of the BKL asymptotics of Einstein's equations near cosmological singularity

Piotr Goldstein and Włodzimierz Piechocki

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We attempt to provide a possibly comprehensive description of the Belinski-Khalatnikov-Lifshitz (BKL) scenario, which depicts an asymptotic of the homogeneous universe near a cosmological singularity. The governing equations, which stem from the Bianchi VIII and Bianchi IX models of spacetime, read

$$\frac{d^2 \ln a}{dt^2} = \frac{b}{a} - a^2, \qquad \frac{d^2 \ln b}{dt^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \qquad \frac{d^2 \ln c}{dt^2} = a^2 - \frac{c}{b}, \tag{1}$$

subject to constraint

$$\frac{d\ln a}{dt}\,\frac{d\ln b}{dt} + \frac{d\ln b}{dt}\,\frac{d\ln c}{dt} + \frac{d\ln c}{dt}\,\frac{d\ln a}{dt} = a^2 + \frac{b}{a} + \frac{c}{b}, \tag{2}$$

where t is the time parameter in the synchronous reference system, while $a = a(t)$, $b = b(t)$ and $c = c(t)$ are the so-called directional scale factors, whose evolution defines the dynamics of the characteristic lengths in three principal directions. The limit $t \to \infty$ corresponds to the cosmological singularity.

Equations (1), (2) have been derived from the Einstein equations by assuming that the anisotropy of space grows without bounds in the evolution towards the singularity [1].

We start from recalling the derivation of equations (1), (2), and the results which we presented at the previous "Integrable Systems" Symposium. These results included the Painlevé analysis, an exact solution, and instability of its small perturbations (absolute perturbations decay but the relative ones grow).

In the present work, we first find that the only Lie symmetries of the equations are translation in time and a simultaneous rescaling of the independent and dependent variables. The self-similar solution with respect to the latter proves to be our exact solution.

The BKL equations can be derived from a Lagrangian, and the constraint (2) requires that corresponding Hamiltonian be zero [2]. By analysing the Hamiltonian structure, we find that a two-dimensional submanifold of initial conditions exists such that the time derivatives of a, b and c remain nonzero as $t \to \infty$, although these scale factors tend to zero. This suggests that they may oscillate with increasing frequency, which is in accordance with the predicted evolution of a small perturbation of our exact solution in the approach to the cosmological singularity [3].

Finally, numerical simulations are performed for testing the chaotic behaviour predicted by BKL [1].

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Mutiple Riemann Waves

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In this talk I will focus on the connection between two seemingly unrelated concepts for solving first-order hyperbolic quasiliear systems of PDEs in many dimensions, namely the symmetry reduction method and the generalized method of characteristics. I will present the outline of recent results on mutiple Riemann wave solutions of these systems. Several examples are included as illustrations of the theoretical results.

On Local Conservation Laws for Generalized Cahn–Hilliard–Kuramoto–Sivashinsky Equation

Pavel Holba

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In this talk we consider nonlinear multidimensional Cahn–Hilliard and Kuramoto–Sivashinsky equations that have many important applications in physics, chemistry, and biology, and a certain natural generalization of these equations.

The generalization in question, the generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation, is a PDE in $n+1$ independent variables $t, x_1 \ldots, x_n$ and one dependent variable u

$$u_t = a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u),$$

Here a is a nonzero constant, b, f, g are smooth functions of u, $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator, $|\nabla u|^2 = \sum_{i=1}^{n}(\partial u/\partial x_i)^2$, and n is an arbitrary natural number.

For an arbitrary natural number n of spatial independent variables we present a complete list of cases when the above PDE admits nontrivial local conservation laws of any order, and for each of those cases we give an explicit form of all the local conservation laws of all orders modulo trivial ones admitted by the equation under study.

In particular, we show that the original Cahn–Hilliard equation and Kuramoto–Sivashinsky equation both admit no nontrivial local conservation laws.

Non-integrability of the planar elliptic restricted three body problem

Andrzej Maciejewski

University of Zielona Góra, Poland

The planar restricted elliptic three body problem is considered. It is proved that for non-zero values of the eccentricity and the mass ratio the system is not integrable. The proof is based on the analysis of the variational equations for the triangular libration point. The same statement is also proved for the parameters' generalization of the system, namely for the planar restricted elliptic photo-gravitational three body problem.

Minimal quantization of Painlevé-type systems

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Linköping University, Sweden

In recent papers we have obtained a way of deforming Stäckel systems into a set of Painlevé-type systems (that systems satisfying Frobenius integrability condition and having isomonodromic Lax representation). In this talk I will first shortly remind this procedure and then I will present minimal quantization of such Painlevé-type systems to sets of non-autonomous, self-adjoint second order operators, acting in an appropriate Hilbert space and satisfying the quantum Frobenius condition (I will call these systems *quantum Painlevé-type systems*), thus guaranteeing that the corresponding Schrödinger equations have common, multi-time solutions. This method leads to two classes of quantum Painlevé-type systems that I will call non-magnetic and magnetic, respectively. I also show the existence of multitime-dependent quantum canonical maps between both these classes of quantum systems. This talk is an effect of collaboration with Maciej Błaszak, Poznań.

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Webs, Nijenhuis operators, and heavenly equations

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In 1989 Mason and Newman proved that there is a 1-1-correspondence between self-dual metrics satisfying Einstein vacuum equation (in complex case or in neutral signature) and pairs of commuting parameter depending vector fields $X(\lambda), Y(\lambda)$ which are divergence free with respect to some volume form. Earlier (in 1975) Plebański showed instances of such vector fields depending of one function of four variables satisfying the so-called I or II Plebański heavenly PDEs. Other PDEs leading to Mason–Newman vector fields are also known in the literature: Husain–Park (1992–94), Konopelchenko–Schief–Szereszewski (2021). In this talk I will discuss these matters in the context of the web theory, i.e. theory of collections of foliations on a manifold, understood from the point of view of Nijenhuis operators. In particular I will show how to apply this theory for constructing new "heavenly" PDEs.

Dark type dynamical systems: their integrability aspects and applications

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In memoriam of our friend and colleague Denis Blackmore (†24.04.2022), who so liked to enlight dark mathematical problems with his light mind.

Some twenty years ago, a new class of nonlinear dynamical systems, called "dark equations" was introduced by Boris Kupershmidt [1], and shown to possess unusual properties that were not particularly well-understood at that time. Later, in related developments, some Burgers-type and also Korteweg-de Vries type dynamical systems were studied [2] in detail, and it was proved that they have a finite number of conservation laws, a linearization and degenerate Lax representations, among other properties. In what follows, we provide a description of a class of self-dual dark-type (or just, dark, for short) nonlinear dynamical systems, which *a priori* allows their quasi-linearization, whose integrability can be effectively studied by means of a geometrically motivated gradient-holonomic approach [3, 4]. Moreover, we study nonlinear dark dynamical systems on functional manifolds which are both of diffusion class and dispersion class and can have interesting applications in modern physics, mechanics, hydrodynamics and biology sciences.

Our report we begin with studying integrability properties of a certain class of nonlinear dynamical systems of the form

$$v_t = K[v, u], \tag{1}$$

on a suitably chosen smooth functional manifold $M_v \subset C(\mathbb{R}; \mathbb{R}^m)$, $m \in \mathbb{N}$, where $t \in \mathbb{R}$ is the evolution parameter and $K : M_v \to T(M_v)$ is a smooth vector field on M_v, with values in its tangent space $T(M_v)$, represented by means of polynomial functions on the related jet-space $J(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m)$ of a finite order, depending *parametrically* on a functional variable $u \in M_u \subset C(\mathbb{R}; \mathbb{R}^n), n \in \mathbb{N}$. Moreover, we will assume that the vector field on M_v satisfies the following additional functional constraint: the flow on

M_v, generated by the vector field (1) possesses an infinite hierarchy of suitably ordered conservation laws (they may be almost all nontrivial, or except finite, trivial), which can be checked by means of solving the corresponding Lax-Noether linear differential-functional equation

$$\varphi_t + K_v^{\prime,*}\varphi = 0 \tag{2}$$

on an element $\varphi \in T(M_v) \otimes \mathbb{C}$, asymptotically depending on a complex parameter $\lambda \in \mathbb{C}$. The latter gives rise to some functional-analytic constraints on the parametric variable $u \in M_u$, which in general can be represented as the following supplementing (1) dynamical system:

$$\begin{aligned} u_t &= F[v,u,p], \\ p_t &= P[v,u,p] \end{aligned} \tag{3}$$

on an extended functional manifold $M_u \times M_p \subset C(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^p)$, where the additional vector fields $F : M_u \to T(M_u)$ and $P : M_p \to T(M_p)$ are smooth and, by definition, finitely-component, that is the functional manifild $M_p \subset C(\mathbb{R};\mathbb{R}^p)$, where the dimension $p \in \mathbb{Z}_+$ has to be finite. The resulting combined dynamical system

$$\left.\begin{aligned} v_t &= K[v,u] \\ u_t &= F[v,u,p] \\ p_t &= P[v,u,p] \end{aligned}\right\} := Q[v,u,p] \tag{4}$$

is closed and determines a completely integrable Lax type linearizable flow on the joint functional manifold $M_v \times M_u \times M_p$. The latter makes it possible to formulate for the dynamical system (1) the following definition.

Definition 1. *A dynamical system (1), allowing the finitely-component and completely integrable Lax-type linearizable extension (4), is called the dark type system.*

This definition proves to be constructive enough and allows by means of the gradient-holonomic approach to classify many linear and nonlinear dynamical systems of dark type, presenting, in addtion, a great interest for applications in modern physics, mechanics, hydrodynamics and biology sciences. To be more specific, we will analyse some of interesting examples of dark type dynamical systems on functional manifolds:

$$\begin{aligned} &v_t = uv_{xx} - uv_x^2, \quad v_t = u^{-2}v_{xx} - u^{-2}v^{-1}v_x^2, v_t = -vv_x - 2u^{-2}v_{xx}, \\ &v_t = -uv_x + v_{xxx}, v_t = v_{xx} - (v^2u)_x, \end{aligned} \tag{5}$$

where the variable $u \in M_u \subset C(\mathbb{R};\mathbb{R})$ is considered to be a functional parameter.

Concluding remarks

We analyzed a new class of nonlinear dark type dynamical systems on functional manifolds, which can be useful for modeling different diffusion and dispersion class processes. Using a gradient-holonomic approach, we were able to prove that these dynamical systems are completely integrable and possessing infinite hierarchies of conservation laws, which generate either finite or infinite number of suitably ordered conservation laws.

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Integrability of Hamiltonian systems with gyroscopic term

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Our main motivation is the integrability analysis of systems given by the following Hamiltonian function

$$H_0 = \frac{1}{2}(p_1^2 + p_2^2) + \omega(p_1 q_2 - p_2 q_1) + V(q_1, q_2), \tag{1}$$

where the potential $V(q_1, q_2)$ is, in general, an algebraic function. The term linear in the momenta is called the gyroscopic or the magnetic term. Sometimes systems with such Hamiltonian are called systems with velocity dependent potentials and are common in physic, astronomy and geometry.

To integrability analysis we prefer the differential Galois framework. This method however needs a non-equilibrium particular solution which is known only in exceptional cases. To overcome this difficulty we use the following idea. First we assumed that the potential $V(q_1, q_2)$ is rational and homogeneous of degree k, and then we performed the classical Levi–Civita regularization. In effect, we obtained system given by Hamiltonian

$$K_0 = \frac{1}{2}\left(v_1^2 + v_2^2\right) + 2\omega(u_1^2 + u_2^2)\left(u_2 v_1 - u_1 v_2\right) + 4(u_1^2 + u_2^2)W\left(u_1, u_2\right) - 4h(u_1^2 + u_2^2), \tag{2}$$

where (u_1, u_2, v_1, v_2) are canonical coordinates, h is a complex parameter,
and $W\left(u_1, u_2\right) = V\left(u_1^2 - u_2^2, 2u_1 u_2\right)$. We prove the following

Theorem 1. *Assume that the system given by Hamiltonian* (2) *satisfies the following conditions:*

1. $V(q_1, q_2)$ *is a homogeneous rational function of degree* $k \in \mathbb{Z}$ *and* $|k| > 2$*,*
2. $V(1, \mathrm{i}) \neq 0$*, or* $V(-1, \mathrm{i}) \neq 0$*,*

then, for arbitrary h*, it does not admit an additional first integral which is a rational function of variable* (u_1, u_2, v_1, v_2) *and is functionally independent with* K_0*.*

However, is spite of this strong and elegant result we cannot conclude that the original system given by Hamiltonian (1) is not integrable.

Nevertheless, the approach proposed above gives the desired result when applied to the family of systems governed by the following Hamiltonian functions

$$H_\mu = \frac{1}{2}(p_1^2 + p_2^2) + \omega(p_1 q_2 - p_2 q_1) - \frac{\mu}{r} + V(q_1, q_2), \tag{3}$$

where the potential $V(q_1, q_2)$ is a rational homogeneous function of degree k, and $r^2 = q_1^2 + q_2^2$. We prove the following result.

Theorem 2. *If the system given by Hamiltonian* (3) *satisfies the following conditions:*

1. $\mu\omega \neq 0$,
2. $V(q_1, q_2)$ *is a homogeneous rational function of degree* $k \in \mathbb{Z}$ *and* $|k| > 2$,
3. $V(1, \mathrm{i}) \neq 0$, *or* $V(-1, \mathrm{i}) \neq 0$,

then it does not admit an additional first integral which is a rational function of variable $(q_1, q_2, p_1, p_2, r, \mu)$ *and is functionally independent with* H_μ.

The possibility of the translation of non-integrability of regularized Hamiltonian systems (2) into non-integrability of these given by (3) follows from the fact that both Hamiltonians (2) and (3) are related to each other by the so-called coupling constant metamorphosis transformation which preserves first integrals.

When knowledge of one integral of motion is sufficient for integrability?

Stefan Rauch

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A system of differential equations is integrable by quadratures when solutions can be expressed using integrations, algebraic operations and taking inverse functions. A general autonomous system of n equations requires knowledge of $n-1$ integrals of motion and one extra integration determines time dependence of solutions. Usually the notion of integrability is associated with Hamiltonian integrability in $2n$ dimensional phase space when knowledge of only n independent and involutive integrals of motion is sufficient for Liouville integrability. This is due to the special nature of the vector-field which is determined by **one function**, the Hamiltonian. But there are known systems of equations when only 2 **or** 1 integral of motion suffice for integrability due to special algebraic form of the equation. There is a trade off between number of integrals and algebraic features of equations.

The purpose of this talk is to make you aware of elegant, little known classes of n 2-nd order Newton equations for which only 1 **quadratic** in velocities integral of motion implies existence of **further** $n-1$ **integrals**. This renders equations integrable and solvable by quadratures through separation of variables. These equations have the triangular form: $d^2 q_r/dt^2 = M_r(q_1, ..., q_r)$ $r = 1, ..., n$ where the r-th equation depends only on the preceding variables q_j, $j = 1, ..., r$.

Recursion operators for multidimensional integrable PDEs

Artur Sergyeyev

Silesian University in Opava, Czech Republic

It is well known that many integrable second-order multidimensional PDEs admit isospectral Lax pairs linear in the spectral parameter and written in terms of first order scalar differential operators. In this talk we present a simple construction of recursion operators for such PDEs.

Stäckel representations of stationary systems of KdV hierarchy and its coupled generalizations

Maciej Błaszak[1], Krzysztof Marciniak[2] and Błażej Szablikowski[1]

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Various invariant reductions of the soliton hierarchies, like stationary flows, lead to the Liouville integrable finite-dimensional systems. The KdV hierarchy is the most researched example of such a type hierarchy. Theory of its stationary flows was developed since the early 70s, when first finite gap solutions of the KdV equation were found by Dubrovin and Novikov and later Bogoyavlenskii and Novikov observed that these flows can be represented by the finite-dimensional Hamiltonian systems. Significant progress was made significantly later when the bi-Hamiltonian formulation for the KdV stationary flows was presented by means of the degenerate Poisson tensors and, in consequence, their Liouville integrability was proved.

Here, we introduce the notion of a stationary system instead of an idea of a single stationary flow and we prove that in the case of the KdV hierarchy such systems have two different Stäckel representations from the Benenti class. These representations are associated with Hamiltonian structures of the KdV hierarchy. Obviously, solutions of these Stäckel systems coincide with the finite gap solutions of the related equations from the KdV hierarchy and hence we come back to the classical results. Besides, what is interesting, the inverse construction is also possible. Actually, starting from the particular family of Stäckel systems one can reconstruct the related hierarchies of stationary systems and hence, in particular, can reconstruct the whole KdV hierarchy.

The above results can be naturally generalized to the coupled KdV hierarchies being multi-Hamiltonian and multi-component analogs of the KdV hierarchy. Their are constructed by means of the so-called energy-dependent Schrödinger spectral problems, which are natural from the point of view of the construction of the spectral curves associated with the stationary systems. In this case each multi-component stationary system has several Stäckel representations from the Benenti class, each associated with different Hamiltonian structure.

The KdV case is presented in the preprint: `arXiv:2204.10632`.

Polyhedra Vibrations

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The polyhedra with exact reflection symmetry group G in the real 3D space is considered. Modifications of the shell of polyhedra that preserve the symmetry are described. Viewing it as a dynamical process, if the change occurs periodically, then it will lead to the vibration of other elements on the shell causing the vibration of the whole polyhedra structure. A simple method of using the symmetry of polyhedra in order to determine several resonant frequencies is presented.

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Double swinging Atwoods machine – from hyperchaos to superintegrability

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We study dynamics and integrability of the double swinging Atwoods machine with additional Hooke's interactions. Complexity of this system is presented with the help of Lyapunov's exponents spectrum's, phase-parametric diagrams and Poincaré cross sections. They show that the system possesses chaotic and hyper-chaotic dynamics, which suggest its non-integrability. We give the analytical proof of this fact via the differential Galois approach and the Kovacic algorithm in dimension four. In the absence of gravity, i.e., $g = 0$ and for certain values of the masses ratio $\mu = M/m$ and the spring constant $k \in R^+$, the considered model is integrable and super-integrable. We find additional first integrals for these cases.

On Hamiltonian structures for WDVV equations

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In this talk we complete establishing that the famous Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations in the case of three and four independent variables are bi-Hamiltonian and provide evidence supporting the conjecture that these equations are bi-Hamiltonian in higher dimensions as well. We also touch upon the computational aspects related to the Hamiltonian structures of WDVV equations. The talk is based on joint research with Raffaele Vitolo.

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Integrable Hamiltonian systems on the symplectic realizations of $\mathbf{e}(3)^*$

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The Lie-Poisson space $\mathbf{e}(3)^* \cong \mathbb{R}^3 \times \mathbb{R}^3$ dual to the Lie algebra $\mathbf{e}(3)$ of the Euclidean group $E(3)$ is the phase space of a heavy top system. We consider the dense open submanifold $\mathbb{R}^3 \times \dot{\mathbb{R}}^3 \subset \mathbf{e}(3)^*$ of $\mathbf{e}(3)^*$ consisting of all 4-dimensional symplectic leaves ($\vec{\Gamma}^2 > 0$) and its two 5-dimensional submanifolds:

(i) submanifold of $\mathbb{R}^3 \times \dot{\mathbb{R}}^3$ defined by $\vec{J} \cdot \vec{\Gamma} = \mu||\vec{\Gamma}||$,

(ii) submanifold of $\mathbb{R}^3 \times \dot{\mathbb{R}}^3$ defined by $\vec{\Gamma}^2 = \nu^2$,

where $(\vec{J}, \vec{\Gamma}) \in \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbf{e}(3)^*$, μ, ν are some real fixed parameters and $\dot{\mathbb{R}}^3 := \mathbb{R}^3 \backslash \{0\}$. Basing on $U(2,2)$-invariant symplectic structure of the Penrose twistor space we find full and complete $E(3)$-equivariant symplectic realizations of these submanifolds. Lifts of the integrable Hamiltonian systems on $\mathbf{e}(3)^*$ (integrable cases of a heavy top) to these symplectic realizations give a large family of integrable Hamiltonian systems.

Angular potentials in superintegrable models on spaces of constant curvature

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A Hamiltonian system with n-degrees of freedom is integrable, in the sense of Liouville, if it admits n global and functionally independent integrals of motion which are in involution. If the integrable system has more global and functional independent integrals of motion than n, it is called superintegrable. The general class of superintegrable systems on 2D spaces of constant curvature can be defined by separated potentials in so-called geodesic polar coordinates. It appears, that there exist two families of potentials of such systems (oscillator and Kepler). The radial potentials are identified depending on the curvature and are uniquely determined. The angular potentials for both families do not depend on the space curvature and are determined implicitly up to an arbitrary function. In this talk, I will construct a new two-parameter family of angular potentials in terms of elementary functions, which reduces to an asymmetric spherical Higgs oscillator. More precisely, I shall first introduce some fundamental notions in the theory of integrable systems. Next, I will present interesting models defined in a 2D space of constant curvature. Then, I shall comment on the superintegrability of such models and introduce the radial and angular actions. Finally, I will discuss an angular potential and describe how to construct a new two-parametric family of angular potentials in terms of elementary functions.

Spectral parameter as a group parameter. Old and new special cases

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The first observation of a general connection between spectral parameters and Lie symmetries is due to Ryu Sasaki [1]. He considered scaling transformations applied to three popular soliton equations (KdV, modKdV and sine-Gordon). In fact, it has long been known that the spectral parameter in the sine-Gordon case (in light-cone coordinates) is related to the Lorentz scaling.

A systematic procedure to produce Lax pairs with a non-removable spectral parameter from non-parametric linear systems by using Lie point symmetries was proposed by Decio Levi and Antoni Sym ([2], [3]) and soon developed in an algebraic form [4]. This method is especially useful for problems motivated by geometry of submanifolds [5, 6]. In the case of inhomogeneous nonlinear Schödinger equation Lie point symmetries turned out to be insufficient and another class (extended Lie symmetries) was needed [7].

We present recent results, including group interpretation of the spectral parameter for several integrable nonlinear equations: hyper-CR equation, reduced quasi-classical self-dual Yang-Mills equation, four-dimensional universal hierarchy equation and the Martínez Alonso-Shabat equation [8]. These equations are characterised by linear systems of two scalar equations.

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